The antiferromagnetic spin- 1/2 -XXZ model on rings with an odd number of sites

1995 J. Phys. A: Math. Gen. 28 4469

(http://iopscience.iop.org/0305-4470/28/16/007)

J. Phys. A: Math. Gen. 28 (1995) 4469–4479. Printed in the UK

The antiferromagnetic spin-$\frac{1}{2}$-XXZ model on rings with an odd number of sites

M Karbach† and K-H Mütter

Physics Department, University of Wuppertal, 42097 Wuppertal, Germany

Received 30 November 1994, in final form 17 May 1995

Abstract. The ground-state energies of the antiferromagnetic XXZ model at a given spin are determined on chains with an odd number N of sites. Analytical and numerical solutions of the Bethe ansatz equations are compared for the N-even and N-odd case. The scaling properties of the ground-state energies enable the determination of the zero-temperature susceptibility. For the isotropic case, we analyse the logarithmic terms in the low-field limit.

1. Introduction

The finite-size behaviour of the energy eigenvalues E_n of the antiferromagnetic XXZ model, with periodic boundary conditions:

$$H := \sum_{x=1}^{N}[S_1(x)S_1(x+1) + S_2(x)S_2(x+1) + \cos\gamma S_3(x)S_3(x+1)] \qquad (1.1)$$

has been studied intensively [1–8] in the critical region $0 \leqslant \rho \equiv \cos\gamma \leqslant 1$. Conformal invariance predicts the finite-size behaviour of the ground and excited states:

$$E_0 = AN - \frac{\pi c}{6N} + \mathrm{O}(N^{-1}) \qquad E_n - E_0 = \frac{2\pi x_n}{N} + \mathrm{O}(N^{-1})$$

where c is the central charge and the x_n are the scaling dimensions of the scaling operators. These predictions have been checked by solving the Bethe ansatz equations analytically for $N \to \infty$ and numerically on finite systems.

The Bethe ansatz equations depend crucially on the quantum numbers of the eigenstate. For example, the momentum of the ground state in the sector with given total spin $S = \sum_x S_3(x)$ is known to be

$$P = -\pi \begin{cases} S & N = 4, 8, 12, \ldots \\ S+1 & N = 2, 6, 10, \ldots \end{cases} \mod 2\pi. \qquad (1.2)$$

For an odd number of sites the ground state is degenerate with two different momenta:

$$P^{(\pm)} = -\pi \begin{cases} S\left(1 \pm \dfrac{1}{N}\right) + \frac{1}{2}(1 \mp 1) & N = 3, 7, 11, \ldots \\ S\left(1 \mp \dfrac{1}{N}\right) + \frac{1}{2}(1 \pm 1) & N = 5, 9, 13, \ldots \end{cases} \mod 2\pi. \qquad (1.3)$$

† E-mail: karbach@wptu10.physik.uni-wuppertal.de

0305-4470/95/164469+11$19.50

Equation (1.3) has been found numerically for $\gamma = 0$ and $N \leqslant 15$ in [8]. In [9] equation (1.3) is shown to be valid for $0 \leqslant \gamma \leqslant \pi/2$.

The Bethe ansatz calculations of [6] were restricted to the N-even case. To our knowledge Alcaraz *et al* [7] were the only ones who solved the Bethe ansatz equations for the N-odd case. It should be noted, however, that our Hamiltonian (1.1) differs from the Hamiltonian used in [7]. The latter is obtained from (1.1), by a minus sign in front of the first two terms and an overall factor of 2.

The outline of the paper is as follows. In section 2 we present the Bethe ansatz equation for the N-odd case. The solution to leading order by means of the Wiener–Hopf method is found in the appendix. The finite-size effects of the ground-state energies at a given total spin S is given in section 3. The scaling properties of these energies are discussed in section 4. They are used to determine the zero-temperature magnetization curve.

2. The Bethe ansatz equations for an odd number of sites

The Bethe ansatz equation for the Hamiltonian (1.1) are well known [2]:

$$\frac{J_i}{N} = \frac{1}{\pi}\tan^{-1}\left(\cot\frac{\gamma}{2}\tanh\frac{z_i}{2}\right) - \frac{1}{N\pi}\sum_{j=1}^{r}\tan^{-1}\left(\cot\gamma\tanh\frac{z_i - z_j}{2}\right) \qquad i = 1, \ldots, r \tag{2.1}$$

with

$$r = \frac{N}{2} - S.$$

For the N-even case, the quantum numbers J_i take the following values in the ground state at a given spin S:

$$J_i = -\frac{r+1}{2} + i \qquad i = 1, \ldots, r = \frac{N}{2} - S. \tag{2.2}$$

This distribution of the J_i is symmetric in the following sense:

$$J_i = -J_{N/2-S+1-i}$$
$$\sum_{i=1}^{N/2-S} J_i = 0.$$

Equations (2.1) and (2.2) represent an implicit equation for the roots z_i from which we can compute the ground-state energies per site, measured from the ferromagnetic state with all spins aligned:

$$e(\gamma, S, N) = -\frac{1}{N}\sum_{i=1}^{N/2-S}\frac{\sin^2\gamma}{\cosh z_i - \cos\gamma}. \tag{2.3}$$

The momentum of the Bethe state is given by the quantum numbers J_i:

$$P = r\cdot\pi - \frac{2\pi}{N}\sum_{i=1}^{r} J_i. \tag{2.4}$$

With the choice (2.2) for the J_i the quantum numbers (1.2) of the ground-state momenta are guaranteed.

In the N-odd case, we start from equations (28) and (29) in the original paper of Bethe [10]. Then one is led to the following choice for the quantum numbers J_i:

$$J_i^{(\pm)} = -\frac{r+1\pm 1}{2} + i \qquad i = 1, \ldots, r = \frac{N}{2} - S. \tag{2.5}$$

This distribution is not symmetric in the sense mentioned above:

$$J_i^{(\pm)} = -J_{N/2-S+1-i}^{(\mp)}$$
$$\sum_{i=1}^{N/2-S} J_i^{(\pm)} = \mp\frac{1}{2}\left(\frac{N}{2} - S\right).$$

Equation (2.5) guarantees that the ground states have the momenta (1.3). We have checked that the choice (2.5) leads to the correct ground-state energy values in case of the isotropic Heisenberg model [8] and the XX model, where the Bethe ansatz equations (2.1) can be solved analytically:

$$z_i = 2\tanh\left\{\tan\frac{\pi}{N}\left[\frac{1}{2}\left(S - \frac{N}{2} - 1 - \delta\right) + i\right]\right\} \qquad \text{for } \gamma = \frac{\pi}{2}$$

with

$$\delta = \begin{cases} \pm 1 & N \text{ odd} \\ 0 & N \text{ even.} \end{cases}$$

Inserting these roots in (2.3) we get for the ground-state energy at a given spin S:

$$e(\gamma = \pi/2, S, N) = -\frac{\cos(\pi\delta/N)\cos(\pi S/N)}{N\sin(\pi/N)}.$$

This is identical to the result obtained by Fabricius [11] by means of a Jordan–Wigner transformation. For $\gamma \neq \pi/2$ the solution of the Bethe ansatz equations (2.1) is more involved. The details of the solution by means of the Wiener–Hopf method are given in appendix A. Here we only quote the result for the ground-state energies at a given S for the N-odd case:

$$e(\gamma, S, N) - e_\infty(\gamma) = -\frac{\pi^2}{12N^2}\frac{\sin\gamma}{\gamma}\left(1 - \frac{3}{2(1-\gamma/\pi)} - 6S^2\left(1 - \frac{\gamma}{\pi}\right)\right) + \mathrm{O}(N^{-4}, N^{-2-4\gamma/(\pi-\gamma)}). \tag{2.6}$$

For convenience we also quote the known result for the N-even case [5, 6]:

$$e(\gamma, S, N) - e_\infty(\gamma) = -\frac{\pi^2}{12N^2}\frac{\sin\gamma}{\gamma}\left(1 - 6S^2\left(1 - \frac{\gamma}{\pi}\right)\right) + \mathrm{O}(N^{-4}, N^{-2-4\gamma/(\pi-\gamma)}). \tag{2.7}$$

(2.6) and (2.7) can be compared with the results of Eggert and Affleck [12], obtained with field theoretical methods. *Identifying in equation (2.45) of [7]* $2\pi R^2 = 1 - \gamma/\pi$ *and* $v = \pi\sin\gamma/(2\gamma)$ one arrives at the expressions (2.6) and (2.7), except for the first term on the right-hand side.

The isotropic case can be obtained in the combined limit:

$$\gamma \to 0 \qquad z \to 0 \qquad z' = \frac{z}{\gamma} \text{ fixed.}$$

In this case the finite-size corrections in next-to-leading order for chains with an odd number of sites turn out to be

$$e(0, S, N) - e_\infty(0) = \frac{\pi^2}{12N^2}\left[\frac{1}{2}\left(1 + \frac{3}{2\ln N}\right) + 6S^2\left(1 - \frac{1}{2\ln N}\right)\right] + \mathrm{O}\left(\frac{\ln\ln N}{N^2(\ln N)^2}, \frac{1}{N^2(\ln N)^2}\right). \tag{2.8}$$

This differs from the result for chains with an even number of sites obtained in [6]:

$$e(0, S, N) - e_\infty(0) = -\frac{\pi^2}{12N^2}\left[1 - 6S^2\left(1 - \frac{1}{2\ln N}\right)\right] + \mathrm{O}\left(\frac{\ln\ln N}{N^2(\ln N)^2}, \frac{1}{N^2(\ln N)^2}\right). \tag{2.9}$$

Following [7] we define

$$c_N(\gamma) \equiv -\frac{12N^2}{\pi^2}\frac{\gamma}{\sin\gamma}(e_0(\gamma, N) - e_\infty(\gamma)) \tag{2.10}$$

where $e_0(\gamma, N)$ is the absolute ground state for chains with an even and odd number of sites, respectively. In the thermodynamical limit we get from equations (2.6)–(2.9):

$$c(\gamma) \equiv \lim_{N\to\infty} c_N(\gamma) = \begin{cases} 1 & N \text{ even} \\ -2\left(1 + \dfrac{3(\gamma/\pi)^2}{4(1-\gamma/\pi)}\right) & N \text{ odd.} \end{cases} \tag{2.11}$$

In the N-even case the ground state has the quantum numbers of the underlying conformal field theory and $c_N(\gamma) = 1$ can therefore be identified with the conformal anomaly. This is not the case for N odd, where the lowest eigenstate has indeed the quantum numbers of an excited state.

Figure 1 shows the ratio $c_N(\gamma)/c(\gamma)$ for chains with an odd and even number of sites as a function of the inverse chain length and various values of the anisotropy.

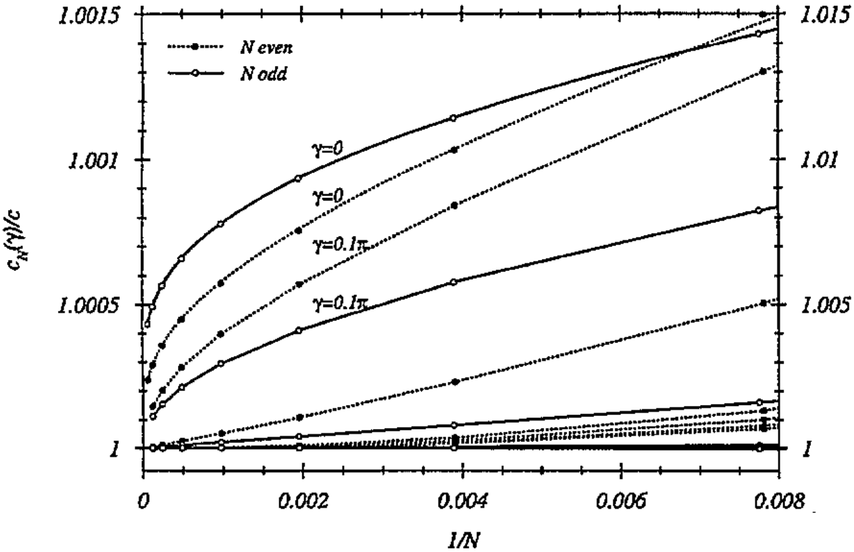

Figure 1. The estimator of the conformal anomaly $c_N(\gamma)$ (equation (2.10)) normalized to the theoretical value $c(\gamma)$ (equation (2.11)) for chains with an even (●) and odd (○) number of sites plotted against $1/N$. The individual curves belong to the anisotropies $\gamma/\pi = 0, 0.1, 0.2, 0.3, 0.4, 0.5$.

The isotropic model is of special interest. Therefore, we discuss this case separately. From equations (2.10) and (2.8), (2.9) we get

$$c_N(0) = a_0 + a_1\frac{1}{\ln^3 N} + a_2\frac{\ln\ln N}{\ln^4 N} + a_3\frac{1}{\ln^4 N} + \cdots \qquad \text{for } N \text{ even} \tag{2.12}$$

$$c_N(0) = b_0 + b_1 \frac{1}{\ln N} + b_2 \frac{\ln \ln N}{\ln^2 N} + b_3 \frac{1}{\ln^2 N} + \cdots \qquad \text{for } N \text{ odd.} \tag{2.13}$$

From the analytic expressions (2.8)–(2.10) and the results of [6] one obtains for the coefficients

$$a_0 = 1 \qquad a_1 = 0.3433\ldots \qquad b_0 = -2 \qquad b_1 = 0. \tag{2.14}$$

On the other hand we have determined these parameters a_i, b_i from a fit to the numerical solutions of the Bethe ansatz on finite rings. These results are listed in tables 1 and 2. They depend on the interval $N_{\min} \leqslant N \leqslant N_{\max}$ for the chain length N.

Table 1. The fits for chains with an even number of sites.

$N_{\min}$–$N_{\max}$	$1-a_0$	a_1	a_2	a_3
128–4096	2.8×10^{-5}	0.545	−1.65	0.81
256–8192	1.4×10^{-5}	0.479	−1.32	0.59
512–16 384	1.2×10^{-5}	0.461	−1.21	0.52
128–4096	0	0.438	−1.16	0.54
256–8192	0	0.401	−0.92	0.33
512–16 384	0	0.374	−0.73	0.15

Table 2. The fits for chains with an odd number of sites.

$N_{\min}$–$N_{\max}$	$2-b_0$	b_1	b_2	b_3
129–4097	4.9×10^{-4}	0.019	−0.28	−0.31
257–8193	6.8×10^{-5}	0.008	−0.24	−0.33
513–16 385	7.4×10^{-5}	0.008	−0.24	−0.33
129–4097	0	0.007	−0.24	−0.32
257–8193	0	0.006	−0.24	−0.33
513–16 385	0	0.009	−0.26	−0.31

The estimated errors are within the last digit. The last three fits start with the theoretical value $a_0 = 1$ and $b_0 = -2$, respectively. From these fits we conclude:

$$c_N(0) = 1 + 0.375(30)\frac{1}{\ln^3 N} - 0.73(40)\frac{\ln\ln N}{\ln^4 N} + 0.15(30)\frac{1}{\ln^4 N} \qquad \text{for } N \text{ even}$$

$$c_N(0) = -2 + 0.008(10)\frac{1}{\ln N} - 0.25(1)\frac{\ln\ln N}{\ln^2 N} - 0.32(1)\frac{1}{\ln^2 N} \qquad \text{for } N \text{ odd.}$$

It is remarkable to note that the two methods lead to consistent results for the coefficient b_0, b_1 in the N-odd case. The situation in the N-even case is different. Here, the analytical result $a_1 = 0.3433\ldots$ obtained by means of the Wiener–Hopf method [6] seems to differ from the numerical result $a_2 = 0.375(30)$. However, it is hard to estimate the error in the numerical result in a reliable way. This discrepancy has been found before by Affleck *et al* [13], who determined a_1 with a third method based on the renormalization group equations. They found $a_1 = 3/8 = 0.375$, which is in good agreement with our numerical result. Finally, let us mention Nomura's [14] result: $a_1 = 0.365\,16(2)$, which was obtained from a numerical solution of the Bethe ansatz equation.

3. Scaling behaviour of the energies and the susceptibility at zero temperature

In this section we are going to discuss the scaling properties of the ground-state energies per site in the various spin sectors:

$$e(\gamma, S, N) = e(\gamma, S/N) + \mathrm{O}(N^{-\alpha}) \qquad \alpha > 0 \qquad 0 \leqslant S/N \leqslant 1/2. \tag{3.1}$$

The validity of (3.1) enables us to compute the magnetization and susceptibility curves at zero temperature, as was shown in [15]. The magnetization at a given external field h:

$$m(\gamma, h(\gamma, S/N)) = \frac{S}{N} \tag{3.2}$$

is defined as the total spin S per site. The magnetization curve then follows from (3.2) and (3.1):

$$h(\gamma, m) = \frac{\partial e(\gamma, m)}{\partial m}.$$

Using the asymptotic expansion (2.6) and (2.7) for the ground-state energies, we find for the low-field behaviour ($h \to 0$) of the magnetization:

$$m(\gamma, h) = \frac{2\gamma}{\pi(\pi - \gamma)\sin\gamma} h(1 + \mathrm{O}(h^2, h^{4\gamma/(\pi-\gamma)})) \qquad 0 < \gamma \leqslant \pi/2. \tag{3.3}$$

In the isotropic case ($\gamma \to 0$) we use the expansions (2.8) and (2.9) and arrive at

$$m(0, h) = \frac{h}{\pi^2}\left(1 - \frac{1}{2}\frac{1}{\ln h} - a_1 \frac{\ln|\ln h|}{\ln^2 h} - a_2 \frac{1}{\ln^2 h} + \cdots\right). \tag{3.4}$$

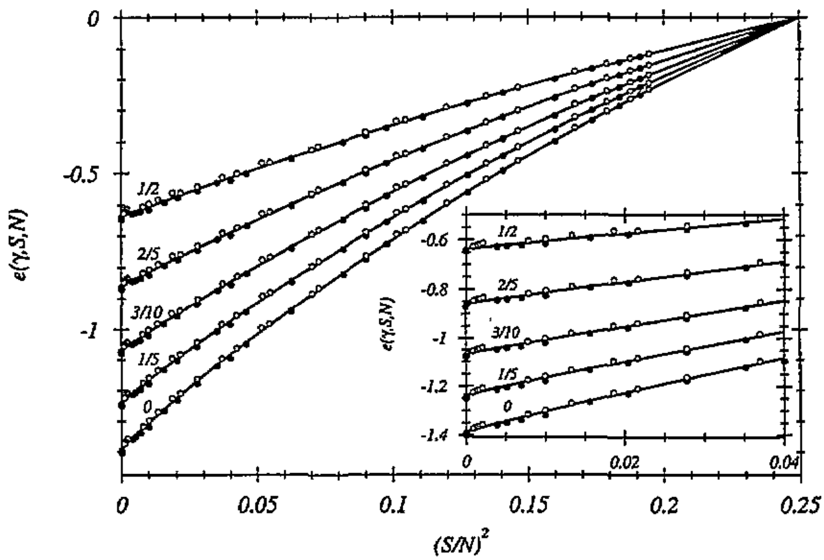

Figure 2. The ground-state energies $e(\gamma, S, N)$ for different anisotropies $\gamma/\pi = 0$, 1/5, 3/10, 2/5, 1/2, plotted against $m^2 = (S/N)^2$ for $N = 11, 13, 15, 17$ (○) and $N = 10, 12, 14, 16$ (●). The full curves are the Bethe ansatz solutions for $N = 2048$.

The scaling property (3.1) is satisfied for all magnetizations and energies and can be seen already on small systems with $N = 10, 12, 14, 16$ and $N = 11, 13, 15, 17$ (see figure 2). However, the results for N odd are systematically above and those for N even below the scaling curve. These tiny deviations can be taken into account by introducing an 'improved' scaling variable:

$$
\begin{aligned}
m_{\mathrm{e}}^2(\gamma, m) &\equiv m^2 - \frac{1-4m^2}{12N^2\zeta(\gamma)} && N \text{ even} \\
m_{\mathrm{o}}^2(\gamma, m) &\equiv m^2 - \frac{1-4m^2}{12N^2\zeta(\gamma)}\left(1 - \frac{3}{4\zeta(\gamma)}\right) && N \text{ odd}
\end{aligned}
\tag{3.5}
$$

where

$$\zeta(\gamma) = \frac{\pi}{2}\frac{\sin\gamma}{\gamma}.$$

The additional term proportional to $1/N^2$ takes into account the lowest-order finite-size corrections of the ground-state energies at a given spin S. It vanishes at $m = 1/2$, where the ground-state energy is given by $e(\gamma, m = 1/2) = 0$ for all N.

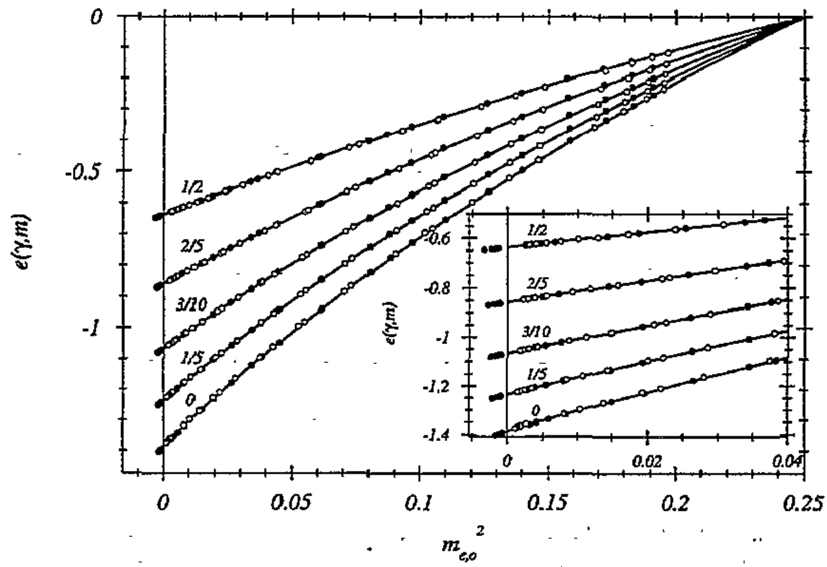

Figure 3. The ground-state energies $e(\gamma, m)$ for different anisotropies $\gamma/\pi = 0$, 1/5, 3/10, 2/5, 1/2, plotted against $m^2_{\mathrm{e,o}}$ (equation (3.5)) for $N = 11$, 13, 15, 17 (○) and $N = 10$, 12, 14, 16 (●). The full curves are the Bethe ansatz solutions for $N = 2048$.

Figure 3 demonstrates that the thermodynamical limit of the scaling curve can be extracted very accurately already from small systems $N \leqslant 17$, once we use the improved scaling variable.

Let us next turn to the zero-temperature susceptibilities

$$\chi(\gamma, h) = \frac{\partial m(\gamma, h)}{\partial h}$$

which can be obtained from finite systems via

$$\chi(\gamma, h) = \frac{1}{N}\frac{1}{h(\gamma, m + 1/N) - h(\gamma, m)}. \tag{3.6}$$

They are shown in figure 4 for various anisotropies. The inset shows a magnification for small fields and $\gamma = 0$. Circles and dots represent Bethe ansatz results from chains

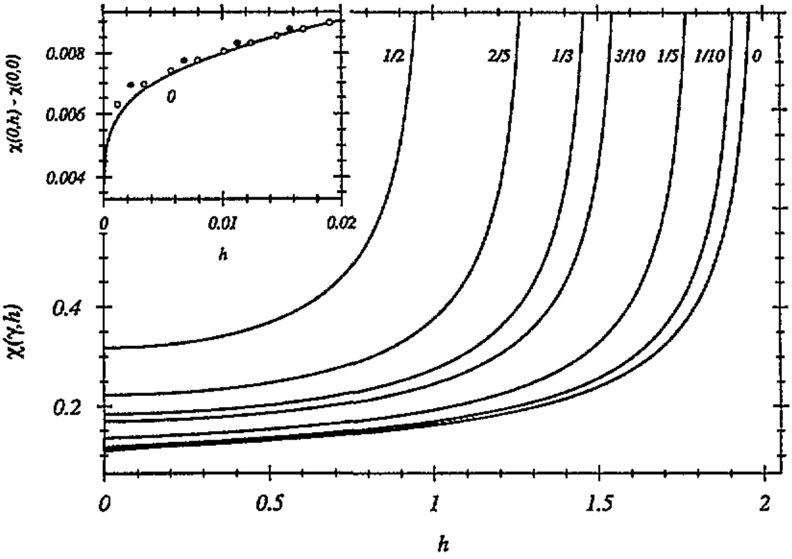

Figure 4. The zero-temperature susceptibility curves $\chi(\gamma, h)$ for different values of γ/π. The inset shows a magnification for small fields and $\gamma = 0$. The circles and dots represent the data points from chains of a length $N = 4096$ (○) and $N = 2048$ (●), respectively. The full curve in the magnification represents the estimates (3.7) to (3.8).

with $N = 2048$ and $N = 4096$, respectively. In the isotropic case the low-field behaviour follows from (3.4):

$$\chi(0, h) = \frac{1}{\pi^2}\left(1 - \frac{1}{2\ln h} - a_1\frac{\ln|\ln h|}{\ln^2 h} - a_2\frac{1}{\ln^2 h} + \cdots\right). \tag{3.7}$$

The coefficient for the higher-order terms a_1, a_2 were estimated from a fit to the numerical results:

$$a_1 = 0.2499(3) \qquad a_2 = 0.25(1). \tag{3.8}$$

Using the Bethe ansatz and the string hypothesis Lee and Schlottmann [16] found $a_1 = 1/4$ which is in good agreement with our results. Finally let us mention that the zero-field susceptibility in the zero-temperature limit ($\mathcal{T} \to 0$) has a similar behaviour to (3.7):

$$\chi(0, h = 0, \mathcal{T}) = \frac{1}{\pi^2}\left(1 - \frac{1}{2\ln\mathcal{T}} + \cdots\right) \tag{3.9}$$

as was found by Eggert *et al* [17] by means of Abelian bosonization. This is easily understood, if the susceptibility scales in the combined limit:

$$\mathcal{T} \to 0 \qquad h \to 0 \qquad z = \frac{h}{\mathcal{T}} \text{ fixed.}$$

Provided that this scaling is correct, we can predict from the low-field behaviour (3.7) the next term in the low-temperature behaviour.

4. Conclusion

The ground-state properties of the XXZ model depend crucially on the number of sites. One has to distinguish four cases:

$$N = 4n + j \qquad j = 0, 1, 2, 3 \qquad n = 1, 2, 3, \ldots.$$

The ground state is unique and translation invariant for $j = 0$. For $j = 2$ the ground state has momentum $p = \pi$. In the N-odd cases ($j = 1, 3$) the ground state turns out to be fourfold degenerate with the two momenta given in (1.3).

In this paper we have determined the ground-state energies at a given total spin for the N-odd case. The finite-size behaviour differs from the N-even case as can be seen from equations (2.6), (2.7) and (2.8), (2.9), respectively. Based on this finite analysis, we studied the scaling behaviour of these ground-states energies. Introducing an improved scaling variable (3.5) we demonstrated that the scaling curve in the thermodynamical limit can be extracted quite accurately from small systems $N \leqslant 17$. From the scaling curve we derived the zero-temperature magnetization and susceptibility. In particular, we studied, for the isotropic case, the logarithmic corrections to the susceptibility in the low-field limit.

Appendix. The Wiener–Hopf solution of the Bethe ansatz equations for N odd

We follow the method introduced by Woynarovich and Eckle [6] in order to analyse the finite-size behaviour of the ground-state energies. Starting from the Euler–MacLaurin formula and the Wiener–Hopf method we solve the resulting integral equations in the standard fashion. In the following, chains with an odd number of sites are considered. We introduce the density of roots for the two distributions $J_i^{(\pm)}$ (equation (2.5)).

$$Z_N^{(\pm)}(z) := \frac{1}{\pi}\tan^{-1}\left(\cot\frac{\gamma}{2}\tanh\frac{z}{2}\right) - \frac{1}{N\pi}\sum_{j=1}^{N/2-S}\tan^{-1}\left(\cot\gamma\tanh\frac{z-z_j}{2}\right)$$

$$\sigma_N^{(\pm)}(z) := \frac{\mathrm{d}Z_N^{(\pm)}(z)}{\mathrm{d}z}.$$

They are related to each other due to (2.5):

$$\sigma_N^{(\pm)}(z) = \sigma_N^{(\mp)}(-z).$$

This leads to the integral representation:

$$\sigma_N^{(\pm)}(z) = \frac{1}{4\gamma\cosh(z\pi/2\gamma)} - \int_{-\infty}^{+\infty} F(z-z')S_N^{(\pm)}(z')\,\mathrm{d}z' \tag{A.1}$$

where

$$F(z) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\omega z)\sinh\omega(\pi-2\gamma)}{2\cosh\omega\gamma\sinh\omega(\pi-\gamma)}\,\mathrm{d}\omega$$

and

$$S_N^{(\pm)}(z) := \frac{1}{N}\sum_{j=1}^{N/2-S}\delta(z-z_j) - \sigma_N^{(\pm)}(z).$$

The energy per site (2.3) is given by

$$e(\gamma, S, N) = e_\infty(\gamma) - \int_{-\infty}^{+\infty}\frac{\pi}{2}\frac{\sin\gamma}{\gamma}\frac{1}{\cosh(z(\pi/2\gamma))}S_N^{(\pm)}(z)\,\mathrm{d}z. \tag{A.2}$$

Here

$$e_\infty(\gamma) = -\sin\gamma\int_0^{+\infty}\left(1 - \frac{\tanh(\omega\gamma)}{\tanh(\omega\pi)}\right)\mathrm{d}\omega$$

is the ground-state energy for the infinite system. Let $\Lambda^{(\pm)}$ denote the largest roots for the two different sets $J_i^{(\pm)}$. They are determined from

$$z_N^{(\pm)}(\Lambda^{(\pm)}) = \frac{1}{2N}\left(\frac{N}{2} - (S+1) \pm 1\right)$$

or, equivalently,

$$\int_{\Lambda^{(-)}}^{\infty} \sigma_N^{(-)}(z)\,\mathrm{d}z + \int_{\Lambda^{(+)}}^{\infty} \sigma_N^{(+)}(z)\,\mathrm{d}z = \frac{1}{N} + \frac{2S}{N}\left(1 - \frac{\gamma}{\pi}\right) \tag{A.3}$$

$$\int_{\Lambda^{(-)}}^{\infty} \sigma_N^{(-)}(z)\,\mathrm{d}z - \int_{\Lambda^{(+)}}^{\infty} \sigma_N^{(+)}(z)\,\mathrm{d}z = \frac{1}{N}. \tag{A.4}$$

Integrals of the type $\int g(z)S_N(z)$ are approximated by means of the Euler–MacLaurin formula:

$$\int_{-\infty}^{+\infty} g(z)S_N^{(\pm)}(z)\,\mathrm{d}z = -\left(\int_{-\infty}^{-\Lambda^{(\mp)}} + \int_{\Lambda^{(\pm)}}^{\infty}\right) g(z)\sigma_N^{(\pm)}(z)\,\mathrm{d}z + \frac{1}{2N}(g(\Lambda^{(\pm)}) + g(-\Lambda^{(\mp)}))$$
$$+\frac{1}{12N^2}\left(\frac{g'(\Lambda^{(\pm)})}{\sigma_N^{(\pm)}(\Lambda^{(\pm)})} - \frac{g'(-\Lambda^{(\mp)})}{\sigma_N^{(\mp)}(-\Lambda^{(\mp)})}\right) + \mathrm{O}\left(\frac{1}{N^3}\right). \tag{A.5}$$

To perform the Wiener–Hopf factorization, we introduce

$$\sigma_N^{(\pm)-} \equiv \begin{cases} \sigma_N^{(\pm)}(z + \Lambda^{(\pm)}) & z > 0 \\ 0 & z \leqslant 0 \end{cases}$$

$$\sigma_N^{(\pm)+} \equiv \begin{cases} 0 & z > 0 \\ \sigma_N^{(\pm)}(z + \Lambda^{(\pm)}) & z \leqslant 0. \end{cases}$$

The Fourier transforms of equations (A.1) and (A.3), (A.4) provide us with a set of equations to determine the leading-order corrections:

$$\hat{\sigma}_N^{(+)}(0) + \hat{\sigma}_N^{(-)}(0) = \frac{1}{2\pi N}\left[1 + 2S\left(1 - \frac{\gamma}{\pi}\right)\right] \tag{A.6}$$

$$\hat{\sigma}_N^{(+)}(0) - \hat{\sigma}_N^{(-)}(0) = \frac{1}{2\pi N} \tag{A.7}$$

$$\hat{\sigma}_N^{(\pm)-}(\omega)[1 - p(\omega)] + \hat{\sigma}_N^{(\pm)+}(\omega) = \frac{\exp \mathrm{i}\omega\Lambda^{(\pm)}}{4\pi \cosh \gamma\omega} + [1 - p(\omega)]$$
$$\times\left[-\frac{1}{4\pi N} + \frac{\mathrm{i}\omega}{24\pi N^2 \sigma^{(\pm)}(\Lambda^{(\pm)})} + \exp \mathrm{i}\omega(\Lambda^{(+)} + \Lambda^{(-)})\right.$$
$$\left.\times\left(\hat{\sigma}_N^{(\pm)-}(-\omega) - \frac{1}{4\pi N} - \frac{\mathrm{i}\omega}{24\pi N^2 \sigma^{(\pm)}(\Lambda^{(\pm)})}\right)\right] \tag{A.8}$$

where

$$p(\omega) = 1 - \frac{\sinh \omega(\pi - 2\gamma)}{2\cosh(\gamma\omega)\sinh \omega(\pi - \gamma)}$$

and

$$\sigma^{(\pm)}(\Lambda^{(\pm)}) = 2\int_{-\infty}^{+\infty} \hat{\sigma}_N^{(\pm)}(\omega)\,\mathrm{d}\omega.$$

Applying the approximation (A.5) on the energy (A.2) yields

$$\begin{aligned} e(\gamma, S, N) - e_\infty(\gamma) = \pi \frac{\sin\gamma}{\gamma} \Bigg[& \exp\left(-\frac{\pi\Lambda^{(-)}}{2\gamma}\right) \\ & \times \left(2\pi\hat{\sigma}_N^{(-)-}\left(\frac{-\mathrm{i}\pi}{2\gamma}\right) - \frac{1}{2N} + \frac{\pi}{24N^2\gamma\sigma_N^{(-)}(\Lambda^{(-)})}\right) + \exp\left(-\frac{\pi\Lambda^{(+)}}{2\gamma}\right) \\ & \times \left(2\pi\hat{\sigma}_N^{(+)-}\left(\frac{-i\pi}{2\gamma}\right) - \frac{1}{2N} + \frac{\pi}{24N^2\gamma\sigma_N^{(+)}(\Lambda^{(+)})}\right)\Bigg] \\ & + \mathrm{O}\left(\mathrm{e}^{-\frac{3}{2\gamma}\pi\Lambda^{(+)}}, \mathrm{e}^{-\frac{3}{2\gamma}\pi\Lambda^{(-)}}\right). \end{aligned} \tag{A.9}$$

Solving the system of equations (A.6)–(A.9) one arrives at the expression (2.6) and (2.8) for the leading finite-size behaviour of the ground-state energies in the N-odd case.

References

[1] Bonner J C and Fisher M E 1964 *Phys. Rev.* A **135** 640
[2] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321, 327; *Phys. Rev.* **151** 258
[3] Cloizeaux J des and Gaudin M 1966 *J. Math. Phys.* **7** 1384
[4] Blöte H W J 1978 *Physica* B **93** 93
[5] Hamer C J 1987 *J. Phys. A: Math. Gen.* **19** 3335
[6] Woynarovich F and Eckle H P 1987 *J. Phys. A: Math. Gen.* **20** L97
[7] Alcaraz F A, Barber M N and Batchelor M T 1988 *Ann. Phys.* **182** 280
[8] Fabricius K, Löw U, Mütter K-H and Ueberholz P 1991 *Phys. Rev.* B **44** 7476
[9] Karbach M 1994 *Thesis* University of Wuppertal WUD 94–6
[10] Bethe H A 1931 *Z. Phys.* **71** 205
[11] Fabricius K 1993 *Z. Phys.* B **92** 519
[12] Eggert S and Affleck I 1992 *Phys. Rev.* B **46** 10866
[13] Affleck I, Gepner D, Schulz H J and Ziman T 1989 *J. Phys. A: Math. Gen.* **22** 511
[14] Nomura K 1993 *Phys. Rev.* B **48** 16814
[15] Fabricius K, Karbach M, Löw U and Mütter K-H 1992 *Phys. Rev.* B **45** 5315
[16] Lee K-J B and Schlottmann P 1987 *Phys. Rev.* B **36** 466
[17] Eggert S, Afleck I and Takahashi M 1994 *Preprint* University of British Columbia UBCTP-94-001